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The computed characteristics of turbulent flow and convection in concentric circular annuli. Part IV: Generalizations

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Abstract

This is the concluding portion of a four-part numerical investigation. The first was limited to the prediction of the velocity field, a prerequisite for the prediction of convection. The second concerned the prediction of the Nusselt number for uniform heating on the inner wall only, and the third the prediction of the Nusselt numbers for most of the other thermal boundary conditions that occur in practice. Herein, generalized predictive equations are presented for all of these conditions. The number of such equations and coefficients is minimized by the use of superposition and generalized expressions for the dependence on *Re* and *Pr*, and to some extent on the aspect ratio.

1. Introduction

This is the fourth and concluding paper on fully developed turbulent flow and fully developed convection in concentric circular annuli. In Part I, Kaneda et al. [1], presented a new, essentially exact, differential model, a set of numerical solutions, and a set of generalized predictive equations for the time-averaged velocity distribution and the mixed-mean velocity. The validity of these results was confirmed by comparisons with the rather extensive although widely scattered prior experimental data and with prior numerically computed values based on heuristic models. In Part II, Yu et al. [2] extended this modeling to convection with uniform heating on the inner wall and an adiabatic outer wall. These conditions correspond to longitudinally and angularly uniform electrical resistance heating of the inner wall as well as to iso-enthalpic counter-current flow through an inner tube and the annulus with perfect insulation on the outer wall. Such conditions may be approached in real heat exchangers away from the entrances.

The validity of this modeling was tested by comparison of the numerically computed values of the Nusselt number with the widely scattered experimental data and with prior numerically computed values based on a questionable model. In Part III, Yu et al. [3] extended the modeling and numerical solutions for most of the other thermal boundary conditions that result in fully developed convection. The results for isothermal surfaces constitute an idealization of heating by a boiling fluid and/or cooling by a condensing fluid. The results

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Nomenclature

a_i	inner radius of annulus (m)
a_o	outer radius of annulus (m)
a_0	radius of zero shear stress (m)
a_{max}	radius of maximum in velocity (m)
a^+	dimensionless radius $[a(\tau_w\rho)^{1/2}/\mu]^+$
A	arbitrary constant
b	half-spacing in parallel-plate channel (m)
b^+	dimensionless half-spacing $[b(\tau_w\rho)^{1/2}/\mu]^+$
B	arbitrary constant
c	specific heat capacity (J/kg K)
f	Fanning friction factor $[2\tau_w/\rho u_m^2]$
h	heat transfer coefficient (W/m^2 K)
k_t	eddy conductivity (W/m K)
k	thermal conductivity (W/m K)
j	radial heat flux density (W/m^2)
m	arbitrary exponent
n	arbitrary exponent
Nu	Nusselt number $[2h(a_o - a_i)/k]$
Nu_0	$Nu\{Pr = 0\}$
Nu_1	$Nu\{Pr = Pr_t\}$
Nu_∞	$Nu\{Pr \to \infty\}$
Nu'_∞	$Nu_\infty\{Pr = Pr_t\}$
Pr	Prandtl number $[c\mu/k]$
Pr_t	turbulent Prandtl number $\left[\frac{Pr(\overline{u'v'})^+(1-(\overline{T'v'})^{++})}{(\overline{T'v'})^{++}(1-(\overline{u'v'})^{++})}\right]$
r	radial coordinate (m)
r^+	dimensionless radius $[r(\tau_{wi}\rho)^{1/2}/\mu]$
R	radius ratio $[r/a_1]$
Re	Reynolds number $[2(a_0 - a_i)u_m/\mu]$
T	time-averaged temperature (K)
T^+	dimensionless temperature $[k(\rho\tau_w)^{1/2}(T_w - T)/\mu j_w]$
T_m	mixed-mean temperature (K)
T'	fluctuating component of temperature (K)
$\overline{T'v'}$	time-average of product of fluctuating temperature and velocity (K m/s)
$(\overline{T'v'})^{++}$	local fraction of radial heat flux density due to turbulence $[\rho c\overline{T'v'}/j]$
u	axial component of time-averaged velocity (m/s)
u^+	dimensionless axial velocity $[u(\rho/\tau_{w1})^{1/2}]$
u_m	mixed-mean axial velocity (m/s)
u'	fluctuating component of axial velocity (m/s)
$\overline{u'v'}$	time-average of product of fluctuating components of velocity (m^2/s^2)
$(\overline{u'v'})^{++}$	local fraction of shear stress due to turbulence $[-\rho\overline{u'v'}/\tau]$
$(\overline{u'v'})^+$	alternative dimensionless shear stress $[-\rho\overline{u'v'}/\tau_{w1}]$
v	radial component of time-averaged velocity (m/s)
v'	fluctuating component of velocity normal to wall (m/s)
x	arbitrary independent variable
y	distance from wall (m)
y^+	dimensionless distance from wall $[y(\rho\tau_w)^{1/2}/\mu]$
z	arbitrary dependent variable; axial coordinate (m)
γ	$[(j/j_{w1})(\tau_{w1}/\tau) - 1]$
ε_o	perturbation of Nu_0 due to convexity (–)
ε_1	perturbation of Nu_1 due to convexity (–)
η_0	perturbation of Nu_0 due to Re (–)
η_1	perturbation of Nu_1 due to Re (–)
μ	dynamic viscosity (Pa s)
μ_t	eddy dynamic viscosity (Pa s)
ρ	specific density (kg/m^3)
τ	shear stress (Pa)
τ_w	shear stress at wall (Pa)

Subscripts

A	adiabatic wall
C	uniformly cooled wall
H	uniformly heated wall
HA	one uniformly heated and one adiabatic wall
HC	one uniformly heated and one uniformly cooled wall
HH	uniformly and equally heated walls
i	of the inner wall
m	mean value
o	of the outer wall
T	isothermal wall
TA	one isothermal and one adiabatic wall
TT	two isothermal walls
w	on the wall
w1	based on shear stress on the inner wall
w2	based on shear stress on the outer wall
wm	based on mean shear stress on the walls
0	for $Pr = 0$
1	for $Pr = Pr_t$
∞	for $Pr \to \infty$

were again validated by comparison with the somewhat limited experimental data and prior numerically computed values of the Nusselt number. The development of predictive equations for these other thermal boundary conditions was deferred to the current presentation in order to take advantage of the generalizations that are described herein. At the loss of some convenience, but in the interests of economy, the analytical developments of Parts I—III are not reproduced herein.

The numerically computed values of *Nu* presented in Parts II and III constitute a sufficient data-base to construct generalized predictive equations for almost all of the thermal boundary conditions that result in fully developed convection, a complete range of values of *Pr*, a range of values of *Re* extending from the minimum for fully turbulent flow to beyond that of any practical interest, and a range of aspect ratios encompassing all values that are likely to be encountered in practice. The only significant missing conditions are for a uniform temperature on one wall with the other insulated and for equal uniform temperatures on both walls for annuli of fractional aspect ratio. Solutions for these latter two conditions aspect ratios appeared to be possible for fractional aspect ratios using the same methodology as for round tubes and parallel-plate channels, but they were not pursued because the computational demands appeared to be excessive relative to the minimal practical importance of the results.

The primary objective of this final paper is to present generalized predictive equations for *Nu* for all of the conditions for which numerical computations have been carried out in this overall investigation. A secondary, but perhaps more important objective has been to examine the concepts and generalizations that led to the numerically computed results and the forms for their representation. Some of these concepts and generalizations are well known, others are often utilized without explicit recognition, while some are new. Their identification is essential to an understanding of the limitations of the models, the numerically computed values, and the predictive equations. This examination might have been included within Parts I–III, but its implications are more obvious in their collective aftermath. The term ‘‘predictive’’ herein implies expressions that are not based directly on experimental data or numerically computed values as distinguished from ‘‘correlative’’ which implies their derivation from such values. The new concepts and generalizations are examined first, and then the resulting generalized predictive equations.

2. Concepts and generalizations

2.1. New concepts and generalizations with respect to flow

The first generalization relevant to this investigation is that of fully developed flow, which is usually implied without recognizing that it is a hypothetical condition, which is only approached asymptotically. For example, deviations from this state are to be expected near the entrance of all channels, depending in character on the configuration of the entrance. The implication of fully developed flow as applied to a heat exchanger is that the channel is of sufficiently length so that the effects of flow development are negligible in an overall sense.

The time-averaged equations of conservation are utilized in this and all other investigations of turbulent flow and convection with the exception of those based on *direct numerical simulation* (DNS). The validity of the concept of time-averaging has been challenged by some, but its wide-spread successful application for the prediction of physical behavior refutes such doubts for all practical purposes.

The first new concept directly related to this investigation of fully developed turbulent flow and convection in annuli was the proposal by Churchill and Chen [4] to model the flow in round tubes, parallel-plate channels, and concentric circular annuli directly in terms of the local turbulent shear stress, as represented by the time-averaged quantity $\rho\overline{u'v'}$, rather than introduce a heuristic model such as the eddy viscosity or the mixing length. Accordingly, they expressed the time-averaged differential momentum balance for a round tube in terms of u^+, y^+, and $(\overline{u'v'})^+ \equiv -\rho(\overline{u'v'})/\tau_w$.

MacLeod [5] postulated that the time-averaged velocity distribution for a round tube in terms of $u^+\{y^+, a^+\}$ is identical to that for a parallel-plate channel in terms of $u^+\{y^+, b^+\}$ if $a^+ = b^+$. This powerful generalization has not generally been recognized in a formal sense although it has occasionally been utilized implicitly. Churchill and Chan inferred from their differential momentum balance in terms of $(\overline{u'v'})^+$ and u^+ that the *analogy* of MacLeod must apply to $(\overline{u'v'})^+$ insofar as it is valid for u^+. They examined the presumably best sets of experimental data for these two dimensionless variables and two geometries and concluded that the analogy and their extension of it were valid, at least within the scatter of those data.

With the advantage of being able to utilize experimental data for both round tubes and parallel-plate channels without discrimination, Churchill and Chan [6] proceeded to construct a correlating equation for $(\overline{u'v'})^+$. In this endeavor they combined theoretically based asymptotes such as $(\overline{u'v'})^+ \sim (y^+)^3$ near the wall, that equivalent to the semi-logarithmic distribution for u^+ in the ‘‘turbulent core near the wall’’, and that equivalent to $u_c^+ - u^+ \sim \left(1 - \frac{y}{a}\right)^2$ near the centerline, in terms of the generalized correlating equation of Churchill and Usagi [7], which consists of arbitrary power-means of pairs of the asymptotes. Although these exponents are empirical, the resulting expression is generally quite insensitive to their numerical value. Since u^+ may be expressed in terms of $(\overline{u'v'})^+$ by direct integration of the differential momentum balance, it follows that a correlating equation for the former can in principle be obtained from one for the latter. However, this integration and its result are too complex for practical purposes. On the other hand, successful and coherent correlating equations in the form of the Churchill–Usagi equation are readily constructed for both u^+ and u_m^+ using integrals of the asymptotes for $(\overline{u'v'})^+$. The reverse process was actually used in Part I for annuli because the experimental data for u^+ were considered to be more reliable than those for $(\overline{u'v'})^+$.

At this point in time, Churchill [8] noted that the alternative dimensionless variable $(\overline{u'v'})^{++} \equiv -\rho(\overline{u'v'})/\tau$, which may be interpreted as the fraction of the local total shear stress due to turbulence, leads to slightly simpler formulations than $(\overline{u'v'})^+$. For example, it was recognized from this alternative formulation that the formal double-integral for u_m^+ that follows from the formal integral for u^+ may be integrated by parts to obtain a single integral with the identical integrand, namely, $1-(\overline{u'v'})^{++}$, although of course with a different factor outside the integral and a different variable of integration, namely, R^2 and R^4, respectively. In retrospect, such an integration by parts was possible for formulations in terms of either the eddy viscosity or the mixing length, but this possibility was apparently never realized because of the greater associated complexity of the integral formulations for these latter quantities. Although the step-wise integration of the differential model to obtain u^+ and u_m^+ proved to be more efficient computationally than evaluation of these integrals by quadrature, the integral formulations are invaluable for the insight they provide concerning the interrelationship between $(\overline{u'v'})^{++}$, u^+, and u_m^+.

The correlating equations for these three dimensionless variables proved to be highly successful in representing experimental data and numerically computed values for round tubes and parallel-plate channels for a complete range of values of y^+ for a^+ and/or $b^+ > 300$, a lower limit imposed by the incorporation of a semi-logarithmic regime for u^+ in the correlating equations. These expressions merit the designation "predictive" because they are only indirectly based directly on experimental data or numerically computed values.

The application of the above concepts, generalizations, and formulations for annuli proved to be uniquely advantageous, but at the same time to result in some complexities not encountered with round tubes and parallel-plate channels. As perhaps first proven by Kjellström and Hedberg [9], the eddy viscosity of annuli is unbounded at one location owing to the finite value of the turbulent shear stress at the location of the maximum in the time-averaged velocity, and negative over an adjacent finite range of the radius. It follows that all numerical solutions based on the eddy viscosity are invalid in principle, although they may produce results of reasonable numerical accuracy owing to insensitivity to the singularity and negative values. Solutions for flow and convection in annuli based on *large eddy simulation* (LES) may be completely unaffected by the singularity and negativity insofar as they do not occur in the region in which the eddy viscosity is utilized, but they are nevertheless subject to error due to the arbitrary expressions used for the eddy diffusivity and/or wall functions in the region near the wall. The quantity $(\overline{u'v'})^+$ is well behaved for all conditions. The quantity $(\overline{u'v'})^{++}$ is itself ill-defined over a short interval but the impact on the computed values of u^+ and u_m^+ appears to be minimal. On the other hand, with the $(\overline{u'v'})^+$ or $(\overline{u'v'})^{++}$ models, additional empiricism beyond that for a round tube or a parallel-plate channel is required in the form of correlating equations for $a_{\min}$, the radial location of the zero in the total shear stress, and $a_{\max}$, the location of the maximum in the time-averaged velocity. Kaneda et al. [1] adapted the predictive equation of Churchill and Chan [6] for u^+ in a round tube to devise separate correlating equations for $(\overline{u'v'})^+$ for the inner region ($a_i \leqslant r \leqslant a_{\max}$) and the outer region ($a_{\max} \leqslant r \leqslant a_o$) of the annulus, and forced the values of both $(\overline{u'v'})^+$ and u^+ to match at $r = a_{\max}$. The numerically computed values for the mixed-mean velocity revealed a surprising and very powerful generalization, namely that the predictive equation for $(u_m^+)_{wm} = (2/f_{wm})^{1/2}$ as a function of Re is invariant for all practical purposes with respect to the aspect ratio a_i/a_o. Fortuitously, this particular friction factor is the one of primary practical interest in that it is based implicitly on the pressure gradient, whereas those based on τ_{w1} and τ_{w2} are applicable only for the indicated wall.

2.2. *New and old concepts and generalizations with respect to the differential energy balance*

The concept of a fully developed temperature field in fully developed flow, and thereby of fully developed convection, is more subtle that that of fully developed flow. It is usually defined as a close approach to an asymptotic value of $(T_w - T)/(T_w - T_m)$, or the equivalent, while T continues to develop radially and longitudinally. Fully developed convection in fully developed flow is usually closely approached in a much shorter channel length than is required for the equivalent approach to fully developed flow. Hence, if the onset of heating occurs at the entrance, flow rather than heat transfer is the limiting process with respect to full development.

Churchill [8] proposed by analogy to that for flow, the expression of the differential energy balance in terms of $T^+ = k(\rho\tau_w)^{1/2}(T_w - T_0)/\mu j_w$ and $(\overline{T'v'})^{++} = \rho c(\overline{T'v'})/j$ rather than introducing the eddy conductivity. The quantity T^+, as well as $(T_w - T)/(T_w - T_m)$, attains a fully developed value for both uniform heating and isothermal heating of one or both walls. The dimensionless variable $(\overline{T'v'})^{++}$, which may be interpreted as the local fraction of the total radial heat flux density due to the turbulent fluctuations, constitutes a completely new approach for turbulent convection in itself. However, $(\overline{T'v'})^{++}$ was promptly replaced by another new variable, namely, $Pr_t/Pr \equiv (\overline{u'v'})^{++}[1-(\overline{T'v'})^{++}]/(\overline{T'v'})^{++}[1-(\overline{u'v'})^{++}]$.

This quantity may be interpreted physically as the ratio of the transport of momentum by the turbulent fluctuations to that by the molecular motions, all divided by the analogous ratio for the transport of energy. The symbol Pr_t and the name *turbulent Prandtl number* were chosen for this ratio of ratios because of its exact equivalence to $c\mu_t/k_t$, where μ_t and k_t are the eddy viscosity and eddy conductivity, respectively. However, the new definition reveals that Pr_t is completely independent of the heuristic diffusional origins of μ_t and k_t.

The advantage of expressing the differential energy balance in terms of Pr_t/Pr rather than $(\overline{T'v'})^{++}$ is twofold. Firstly, the variance of $(\overline{T'v'})^{++}$ on location and the rate of flow is mostly subsumed by that of $(\overline{u'v'})^{++}$, an already a known quantity. Secondly, the dependency of the quantity Pr_t itself is remarkably constrained. Abbrecht and Churchill [10] asserted in 1960, on theoretical grounds as well as on the basis of their own experimental data for developing convection, that Pr_t is independent of the thermal boundary condition(s) and geometry, and thereby a function only of $(\overline{u'v'})^{++}$ and Pr, at least in the turbulent core. This postulate is one of the most helpful generalization of all those mentioned to this point. Although it has apparently never been formally proven or disproven, nor have its possible limits of applicability, if any, been established, this independence is an implicit in almost all the analyses that have ever been carried out for turbulent convection. Furthermore, the dependence of Pr_t on $(\overline{u'v'})^{++}$ is at most second-order as attested by correlations of experimental data such as that of Jischa and Reike [11] for Pr_t as a function only of Pr, and in any event the effect of such a dependence on Nu is negligible as attested by the comparative numerical calculations by Yu et al. [12] using correlating equations for Pr_t that both included and omitted a dependence on $(\overline{u'v'})^{++}$. This matter has been given such detailed consideration because it proves to be critical to the development of the correlating equations to be presented for Nu.

2.3. Generalization of the dependence of Nu on Pr/Pr_t

Churchill [8], as well as introducing the dimensionless quantities $(\overline{u'v'})^{++}$ and $(\overline{T'v'})^{++}$ and expressing Pr_t/Pr in terms of only these two quantities, developed asymptotic integral expressions for $Nu_0 \equiv Nu\{Pr = 0\}$ and $Nu_1 \equiv Nu\{Pr = Pr_t\}$, and an algebraic one for $Nu_\infty \equiv Nu\{Pr \to \infty\}$ as possible components of a correlative equation for $Nu\{Re, Pr\}$. However, before devising such a correlative equation a better alternative was discovered as follows. In the course of a critical analysis of the classical analogies between momentum and energy transfer, Churchill [13], combined the partially algebraic, partially numerical, and partially graphical fragments of the analogy of Reichardt [14] to obtain a single algebraic equation. Churchill et al. [15] subsequently recognized that this reformulation could be interpreted as a simple algebraic combination of Nu_1, Nu_∞, and Pr_t/Pr, namely,

$$\frac{1}{Nu} = \left(\frac{Pr_t}{Pr}\right)\frac{1}{Nu_1} + \left(1 - \frac{Pr_t}{Pr}\right)\frac{1}{Nu_\infty}. \quad (1)$$

Without this previous development of the detailed asymptotic formulations for Nu, the identification of this simplified form of the Reichardt analogy would probably not have occurred. Eq. (1) may be recognized to be free of explicit empiricism, but, even so, it is presumably not exact owing to the several idealizations made by Reichardt in order to be able to integrate the combination of the differential momentum and energy balances in closed form. One of his idealizations was the afore-discussed postulate of the independence of Pr_t from location; the other approximations were purely mathematical in character. Eq. (1) is applicable only for $Pr \geqslant Pr_t$ by virtue of these latter idealizations, but Churchill et al. [15] derived an analogue for $Pr \leqslant Pr_t$ in terms of Nu_0, Nu_1, Pr_t/Pr, and $Nu^1_\infty \equiv Nu_\infty\{Pr = Pr_t\}$. This latter term arises from matching the derivatives of the expressions for $Pr \leqslant Pr_t$ and $Pr \geqslant Pr_t$ at $Pr = Pr_t$. The expressions for Nu_0, Nu_1, Nu_∞, and Nu^1_∞ each incorporate a dependence on Re. Hence, Eq. (1) and its analogue for $Pr \leqslant Pr_t$, predict the dependence on Re implicitly as well as that on Pr/Pr_t explicitly.

Eq. (1) and its analogue for $Pr \leqslant Pr_t$ are revolutionary in concept in that they purport to replace the classical purely empirical correlating equations of limited range in the form of products of powers of Re and Pr with two bilinear algebraic equations with a theoretically based structure, no explicit empiricism, and, in combination, all values of Pr and all values of Re in the regime of fully developed turbulence. These two expressions were shown graphically to represent numerically computed values for round tubes and parallel-plate channels for all values of Pr/Pr_t, all values of Re in the regime of fully developed turbulence, and all thermal boundary conditions almost perfectly in visual terms, and certainly far better than any prior expressions. As an example of their functional superiority, they predict a point of inflection in the dependence of Nu on Pr/Pr_t in the low range of that variable that is clearly confirmed in retrospect by experimental data even though it has apparently never been remarked on, and a second one at moderate values of Pr/Pr_t that is obscured by the scatter of experimental data but whose existence is required by the theoretically based dependence incorporated in the asymptotes.

Despite this overall success, small numerical discrepancies in the predictions were noted at two particular values of Pr/Pr_t. Churchill and Zajic [16] deduced that these discrepancies were a direct consequence of the mathematically based idealizations made by Reichardt. In the process of trying to improve upon these idealizations they discovered that an analogy of Churchill [17], previously under-rated even by its author, when re-expressed in terms of Nu_1, Nu_∞, and Pr/Pr_t, accomplished that objective. That expression and its analogue for $Pr \leqslant Pr_t$ were presented in Part II in a slightly improved form as obtained by rearrangement.

Direct comparisons of predictions of Nu with experimental data are difficult because of the irregular values of the several parameters such as Pr, the mode of heating, and the sign and magnitude of the temperature difference. Hence, Churchill and Zajic compared the predictions of their equations as well those of a selection of the most widely used correlating equations of the past for Nu in round tubes, including some that share part of the structure of their own expressions, with a correlating equation of Churchill [18] that represents the culled data for small temperature-differences for all conditions remarkably well by means of two expressions in the form of the Churchill–Usagi equation that incorporate nine arbitrary coefficients and seven arbitrary exponents. The predictions of the two equations of Churchill and Zajic, which incorporate no explicit empiricism other than one exponent, were shown to agree almost exactly with this empirical representation of experimental data. This great functional as well as numerical improvement is a primarily a consequence of the recognition that the analogy of Reichardt for energy and momentum transfer, when re-formulated in terms of asymptotic expressions for Nu, obviates the need for a correlating equation. This indirect comparison with the experimental data is a necessary but of course not a sufficient test of accuracy because of the considerable scatter of the experimental data. Accordingly a further test in terms of parametric sensitivity has been carried out by Churchill et al. [19].

The predictive equations of Churchill and Zajic, together with the numerically computed values of Nu_0 and Nu_1 and a theoretical expression for Nu^1_∞, have already been demonstrated graphically in Parts II and III to represent the computed values of Nu for annuli for all thermal boundary conditions, all values of the aspect ratio a_i/a_o, and all values of $(a_o^+ - a_i^+)_{wi}$ almost perfectly as a function of Pr/Pr_t. There is, however, a price to be paid for the improvement in numerical accuracy, functional accuracy, and scope associated with these predictive equations. If they are to be used to predict Nu for specified values of Re, Pr, and a_i/a_o, it is necessary to have supplementary correlating equations for Nu_0 and Nu_1 as a function of Re and a_i/a_o for each thermal boundary condition, as well as one for Pr_t as a function of Pr. Furthermore, since Nu_0 and Nu_1 are based on the difference between the temperature of one of the walls and the mixed-mean temperature of the fluid, separate values are required for each wall that is heated or cooled. The development and testing of such supplemental expressions is described in the following two subsections.

2.4. Generalization of the dependence of Nu_0 and Nu_1 on Re and a_o/a_i

The numerical calculations for Nu were carried out for a series of fixed values of $(a_o^+ - a_i^+)_{wi}$ and a_i/a_o. However, the correlating equations for Nu_0 and Nu_1 for each heated or cooled surface would normally be applied for fixed values of $Re = 2(u_m^+)_{wm}(a_o^+ - a_i^+)_{wi}(\tau_{wm}/\tau_{wi})^{1/2}$, and on theoretical grounds those for Nu_1 in terms of $Nu_{1i}/Re(f_{wi}/2)$ and $Nu_{1o}/Re(f_{wo}/2)$. Hence it is necessary to be able to interrelate Re, $(a_o^+ - a_i^+)_{wi}$, τ_{wo}/τ_{wi}, τ_{wm}/τ_{wi}, $(u_m^+)_{wm} = (f_{wm}/2)^{1/2}$, f_{wi}, and f_{wo}. A set of algebraic equations for these various quantities for all conditions is developed in Part I [1] as well as a tabulation of a sufficient set of values for these variables for the conditions of the numerical computations.

Without any justification other than simplicity, the correlating equations for Nu_0 and $Nu_1/Re(f/2)$ were formulated as the product of two terms, one for the second-order dependence on Re and the other for the dependence on a_i/a_o. This allowed a correlative equation to be devised for the dependence on Re for $a_i/a_0 = 1.0$, and then one for the dependence on a_1/a. Very simple expressions of the generic form $z = [A + Bx^n]^m$, usually with $m = 1$, were utilized for both of these components. Such a simple scheme proved satisfactory because of the limited ranges of variation of Nu_0 and $Nu_1/Re(f/2)$. The second-order dependence of both Nu_0 and Nu_1 on Re was actually expressed in terms of $(u_m^+)_{wm}$, because it has a more limited range of values.

2.5. Generalization of Nu for different thermal boundary conditions

Superposition is a well-known mathematical tool that is applicable to convective heat transfer insofar as the equation for the conservation of energy is linear in temperature. A linear dependence on temperature is inherent in fully developed laminar convection insofar as the physical properties are invariant and viscous dissipation is negligible, and superposition has often been applied. Superposition has also been applied for fully developed turbulent convection, but often without careful assessment of the restriction of linearity, in particular with respect to the possible dependence of the turbulent Prandtl number on the temperature field. Kays [20] concludes, on the basis of two independent sets of experimental data, and despite the compelling contrary evidence of Abbrecht and Churchill [10], that the turbulent Prandtl number is different for uniform and isothermal heating. However, this does not preclude the application of superposition insofar as those conditions are treated separately. In any event, the model used for the numerical computations whose results are examined herein was linear in temperature, so superposition is applicable insofar as the model is valid.

The simplest form of superposition is for the overall Nusselt number based on the total heat input and the difference between the arithmetic average of the wall temperatures and the mixed-mean temperature, namely,

$$Nu_m \equiv \frac{2(a_o - a_i)(j_i a_i + j_{2o} a_o)}{k(a_o + a_i)\left[\frac{(T_{wi}+T_{wo})}{2} - T_m\right]} = \frac{2j_{ave}}{\frac{j_i}{Nu_i} + \frac{j_o}{Nu_o}}. \quad (2)$$

Here, $j_{ave} = (j_i a_i + j_o a_0)/(a_i + a_o)$, j_i is the heat flux density on the inner surface of radius a_i, j_o is the heat flux density on the outer surface of radius a_o, Nu_i is the Nusselt number for heat transfer to the fluid based on the heat flux density j_i from that wall to the fluid and the difference between the temperature of the outer wall and the mixed-mean temperature of the fluid, namely, $Nu_i \equiv \frac{2(a_o-a_i)j_i}{k(T_i-T_m)}$, and $Nu_o \equiv \frac{2(a_o-a_i)j_o}{k(T_o-T_m)}$ is its analog for the outer wall. Nu_i is influenced by the heat flux, if any, to or from the outer wall because of the influence of the latter on T_m, and Nu_o is similarly influenced by the heat flux, if any, on the inner wall. Attention herein is focused on Nu_i and Nu_o rather than on Nu_m because the coefficients for a single surface are more simply related to the fluid mechanics, the aspect ratio, and the thermal boundary conditions.

The simplest but most important expression for the application of superposition for an annulus is

$$Nu_i = \frac{Nu_{iHA}}{1 - \alpha\left(\frac{j_o a_o}{j_i a_i}\right)}. \quad (3)$$

Here, the subscript HA designates uniform heating on one wall only. Eq. (3) is presumed on the basis of its derivation to be applicable for all values of $j_o a_o / j_i a_i$ and therefore may be considered to generalize a solution for one set of boundary conditions for all others of the same class, such as those involving uniform heat fluxes on the two surfaces. With the subscripts o and i, inverted, Eq. (3) is applicable for the outer wall. For fully developed laminar convection, the derivation of an exact numerical value or theoretical expression for the *influence coefficient* α that is independent of *Re*. and of course *Pr* may be possible. For example, for uniformly heated parallel plates, $\alpha = 7/26$. Eq. (3) is also applicable for fully developed turbulent convection but then α is a function of *Re*, *Pr*, and a_i/a_o. However it can be applied for specific values of *Pr*, and therefore may be applied separately for Nu_0 and Nu_1.

Numerical values of α can be determined from numerically computed values of *Nu* for two different modes of heating, that is for two different values of $j_o a_o / j_i a_i$, for otherwise identical conditions. It follows that numerical values of α may be calculated from any numerical pair of values of the quantities Nu_{HA}, Nu_{HH}, and Nu_{HC}, and in turn the third of these three. Here the subscripts HH designates uniform equal heat fluxes on the two surfaces, and HC uniform equal heating and cooling. Some useful relationships between α, Nu_{HA}, Nu_{HH}, and Nu_{HC} are listed in Table 1 together with expressions for *Nu* as a function of Nu_{HA}, Nu_{HH}, Nu_{HC}, and $j_o a_o / j_i a_i$. From these expressions, it is apparent that numerical values or correlating equations for any two quantities such as α, Nu_{HA}, Nu_{HH}, and Nu_{HC}, are sufficient to predict Nu_i or Nu_o for any combination of uniform heating and cooling on the two surfaces of an annulus. The choice of α as one of the two variables for tabulation and correlation appears to be advantageous on the basis of its more constrained range of values. Kays and Leung [21], in the only extensive prior numerical investigation of turbulent convection in annuli, chose Nu_{HA} as the second variable, but Nu_{HH}, and Nu_{HC} both appear to have equal merit.

The influence coefficient α, as well as Nu_{HA}, Nu_{HH}, and Nu_{HC}, are functions of the same variables as *Nu*, that is of a_1/a_2, $(a_2^+ - a_1)_{w1}$, the direction of heat transfer (inward or outward), and *Pr*. However, as mentioned in connection with *Nu*, only the values of α, Nu_{HA}, Nu_{HH}, and/or Nu_{HC} for $Pr = 0$ and Pr_t need be determined. Correlating equations for α, Nu_{HA}, and Nu_{HH}, as well as tests of their success follow.

2.6. Correlating equations for uniform heating and cooling and their evaluation

On the basis of superposition and the framework of the predictive equations of Churchill and Zajic, only eight correlating equations, four for each surface, are necessary for a complete representation for all

Table 1
Useful relationships for superposition

$$Nu_{HH} = \frac{Nu_{HA}}{1-\alpha}, \quad Nu_{HC} = \frac{Nu_{HA}}{1+\alpha}, \quad Nu_{HC} = \left(\frac{1-\alpha}{1+\alpha}\right)Nu_{HH}$$

$$\alpha = 1 - \frac{Nu_{HA}}{Nu_{HH}} = \frac{Nu_{HA}}{Nu_{HC}} - 1 = \frac{Nu_{HH} - Nu_{HC}}{Nu_{HH} + Nu_{HC}}$$

$$\frac{1}{Nu} = \left(1 + \frac{j_o a_o}{j_i a_i}\right)\frac{1}{Nu_{HA}} - \left(\frac{j_2 a_o}{j_1 a_i}\right)\frac{1}{Nu_{HC}} = \frac{i}{Nu_{HA}} - \frac{j_o a_o}{j_i a_i}\left(\frac{i}{Nu_{HC}} - \frac{1}{Nu_{HA}}\right)$$

$$\frac{2}{Nu} = \left(1 + \frac{j_o a_o}{j_i a_i}\right)\frac{1}{Nu_{HH}} - \left(1 - \frac{j_o a_o}{j_i a_i}\right)\frac{1}{Nu_{HC}} = -\left(\frac{1}{Nu_{HC}} - \frac{1}{Nu_{HH}}\right) + \frac{j_o a_o}{j_i a_i}\left(\frac{1}{Nu_{HC}} + \frac{1}{Nu_{HH}}\right)$$

combinations of uniform heating and cooling of annuli, for example, α_{0iHA}, α_{1iHA}, α_{0oHA}, α_{1oHA}, Nu_{0iHA}, Nu_{1iHA}, Nu_{0oHA}, and Nu_{1oHA}. Correlating equations for these particular eight quantities are presented in Table 2, along with expressions for the only other quantities or their equivalents that occur in the these eight correlating equations, namely $(u_m^+)_{wm}$ as a function of $(a_o^+ - a_i^+)_{wn}$, and τ_{wm}/τ_{wi}, τ_{wm}/τ_{wo}, and $(u_m^+)_{wm}$ as functions of a_i/a_o.

Tests of the individual success of the correlating equations in reproducing the numerically computed values are presented in Tables 3–10, and of their combined success in reproducing the numerically computed values of Nu_{0iHH}, Nu_{1iHH}, Nu_{0oHH}, Nu_{1oHH}, Nu_{0iHC}, Nu_{0oHC}, Nu_{0iHC}, and Nu_{1iHC} in Tables 11–18. Somewhat simpler expressions of equivalent accuracy in place of the combination of correlations for Nu_{HA} and α could undoubtedly be constructed for each of these latter 8 quantities, but the lesser number of correlating equations appears to more than compensate.

All of the numerically computed values of Nu except for a few bordering conditions for which the results were suspect are tabulated in Parts II and III. The missing values correspond to one or more of the following: (1) the computed value of Re was less than the presumed minimum value for fully developed turbulence, (2) the derived values of the matching coefficient of the two parts of the differential model were suspect in value, or (3) the convergence of the numerical computations for the time-averaged velocity distribution with grid-size was suspect. The comparisons of the computed and predicted values of Nu in Tables 3–18 are, in the interests of avoiding unnecessary detail, for arbitrarily selected values of a_i/a_o and selected values of $(a_o^+ - a_i^+)_{wi}$ in the range from 500 to 20,000. This lower limit corresponds roughly to the value of 300, below which the semi-logarithmic expression for the velocity distribution, which effects the correlating equations for $(\overline{u'v'})^+$, no longer has any region of validity, while 20,000 represents the approximate upper limit of practical interest, corresponding to $Re \approx 10^6$. The predictive equations may be reasonably accurate for some of the missing conditions in Tables 3–18, or for $a_i/a_o < 0.01$ but there is no reliable criterion for their evaluation for these conditions. The values predicted by the correlating equations are enclosed in parentheses for identification. The overall accuracy of the predictions is satisfactory. That for Nu_0 is generally better than for Nu_1, primarily because of its more limited range in magnitude. Significant deviations are identified by bold face, which is applied to both the computed and predicted values because in some cases the computed values appear to violate trends with a_i/a_o and $(a_o^+ - a_i^+)_{wi}$, and therefore may be the source of the discrepancy. As would be expected, the questionable values generally occur for the same conditions in all of the tables.

Table 2
Correlating equations for Nu_0, Nu_1, and α

$$Nu_{0iHA} = \frac{5.872\left(1+\frac{0.08}{(a_i/a_o)^2}\right)^{1/3}}{1+\frac{0.8}{(u_m^+)_{wm}}}$$

$$Nu_{0oHA} = 5.568\left[1+0.002(u_m^+)_{wm}\right]\left[1-072\left(1-\frac{a_i}{a_o}\right)^{4/3}\right]$$

$$Nu_{1iHA} = \frac{Re\left(1+0.288(a_i/a_o)^{0.28}\right)}{1.288(u_m^+)^2_{wm}\left(\frac{\tau_{wm}}{\tau_{wi}}\right)\left(1+\left(\frac{53.3}{(u_m^+)_{wm}}\right)^3\right)^{1/10.6}}$$

$$Nu_{1oHA} = \frac{Re\left(1+0.152\left(1-\frac{a_i}{a_o}\right)^4\right)}{(u_m^+)^2_{wm}\left(\frac{\tau_{wm}}{\tau_{wo}}\right)\left(1+\left(\frac{53.3}{(u_m^+)_{wm}}\right)^3\right)^{1/10.6}}$$

$$\alpha_{0i} = \left(0.3922+\frac{(u_m^+)_{wm}}{358.6}\right)\left(0.162+0.838\left(\frac{a_i}{a_o}\right)^{1/3}\right)$$

$$\alpha_{1i} = \left(0.0643+\frac{2.745}{(u_m^+)_{wm}}\right)\left(0.237+0.763\left(\frac{a_i}{a_o}\right)^{1/2}\right)$$

$$\alpha_{0o} = \left(0.3922+\frac{(u_m^+)_{wm}}{358.6}\right)\left(1.64-0.64\left(\frac{a_i}{a_o}\right)^{7/8}\right)$$

$$\alpha_{1o} = \left(0.0643+\frac{2.745}{(u_m^+)_{wm}}\right)\left(1+1.180\left(1-\frac{a_i}{a_o}\right)^{1.5}\right)$$

$$Nu_{0TT} = \frac{12}{1+\frac{1.08}{(u_m^+)^{0.394}}}$$

$$Nu_{1TT} = \frac{Re(f/2)}{\left(1+\frac{110}{(u_m^+)^{5/2}}\right)} = \frac{4b^+}{u_m^+\left(1+\frac{110}{(u_m^+)^{5/2}}\right)}$$

$$\alpha_{0T} = 0.4076+0.00225u_m^+, \qquad \alpha_{1T} = 0.0675+\frac{3.012}{u_m^+}$$

Supplementary expressions and notes:

(1) $(u_m^+)_{wm} = 3.2+\frac{1}{0.436}\ln\left\{(a_2^+ - a_1^+)_{wm}\right\} - \frac{275}{(a_2^+ - a_1^+)_{wm}}$.

This expression was devised in terms of u_m^+ and a^+ for a round tube, but was found to provide a very good approximation for $(u_m^+)_{wm}$ for all values of a_i/a_o. It is applicable in terms of Re and f_{wm} as well because $(u_m^+)_{wm} = \left(\frac{2}{f_{wm}}\right)^{1/2}$ and $Re = 2(a_0^+ - a_i^+)_{wm}(u_m^+)_{wm}$.

(2) $\frac{\tau_{wm}}{\tau_{wi}} = \left(\frac{a_o}{a_i}-1\right)\Big/\left[\left(\frac{a_0}{a_i}\right)^2-1\right]$ and $\frac{\tau_{wm}}{\tau_{wo}} = \frac{a_o}{a_i}\left(\frac{a_o}{a_i}-1\right)\Big/\left[\left(\frac{a_o}{a_i}\right)^2-\left(\frac{a_0}{a_i}\right)^2\right]$, with $\frac{a_0}{a_i} = \frac{1+\frac{a_o}{a_i}\left(\frac{a_o}{a_i}\right)^{0.386}}{1+\left(\frac{a_o}{a_i}\right)^{0.386}}$.

These expressions for τ_{wm}/τ_{wi} and τ_{wm}/τ_{wo} are exact but that for a_0/a_i has empirical roots (see [1]).

Table 3
Computed and predicted values of Nu_{0iHA}

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o						
	0.01	0.05	0.1	0.2	0.5	0.8	0.999
800		17.20 (18.01)	11.55 (11.69)	8.324 (8.107)	6.303 (6.168)	5.874 (5.849)	5.772 (5.771)
1000	52.28 (52.19)	17.20 (18.04)	11.55 (11.71)	8.345 (8.118)	6.310 (6.177)	5.884 (5.875)	5.784 (5.779)
5000		17.19 (18.21)	11.54 (10.85)	8.351 (8.178)	6.337 (6.221)	5.927 (5.899)	5.853 (5.820)
20,000			11.53 (11.84)	8.348 (8.213)	6.348 (6.248)	5.946 (5.924)	5.860 (5.845)

Table 4
Comparison of predicted and computed values of α_{0i}

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o					
	0.01	0.05	0.1	0.2	0.5	0.999
800		0.2150 (0.2077)	0.2485 (0.2434)	0.2882 (0.2884)	0.3549 (0.3661)	0.4429 (0.4428)
1000	0.1535 (0.1512)	0.2153 (0.2085)	0.2495 (0.2443)	0.2892 (0.2894)	0.3565 (0.3675)	0.4454 (0.4444)
5000		0.2176 (0.2139)	0.2526 (0.2503)	0.2929 (0.2969)	0.3632 (0.3766)	0.4582 (0.4553)
20,000			0.2537 (0.2554)	0.2943 (0.3025)	0.3658 (0.3840)	0.4644 (0.4642)

Table 5
Comparison of computed and predicted values of Nu_{0iHH}

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o					
	0.01	0.05	0.1	0.2	0.5	0.999
800		21.91 (22.81)	15.37 (15.52)	11.72 (11.50)	9.770 (9.875)	10.36 (10.36)
1000	61.77 (61.70)	21.92 (22.84)	15.39 (15.54)	11.74 (11.52)	9.805 (9.903)	10.43 (10.40)
5000		21.97 (23.01)	15.44 (15.68)	11.81 (11.66)	9.951 (10.08)	10.77 (10.69)
20,000			15.45 (16.79)	11.83 (11.77)	10.01 (10.22)	10.94 (10.91)

Because of the choice of such a simple form for correlation in the interests of convenience and transparency, the application of the correlating equations for Nu_{HA} and α is not recommended for prediction for values of a_1/a_2 below 0.01, which is the lower limit of the numerical computations, or for $Re > 10^6$ which is their upper limit, but these are not serious restrictions from a practical point of view.

2.7. Generalities with respect to the correlating equations for uniform heating and cooling

Several generalities concerning the above correlating equations may be inferred from the equations summarized in Table 2 or from physical considerations. For example, it may be noted that $Nu_{0iHC} = 4.0$ corresponds to pure thermal conduction because as $Pr \rightarrow 0$,

Table 6
Computed and predicted values of Nu_{0iHC}

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o					
	0.01	0.05	0.1	0.2	0.5	0.999
800		14.16 (14.96)	9.243 (9.443)	6.457 (6.354)	4.652 (4.581)	4.001 (4.000)
1000	45.33 (45.49)	14.15 (14.96)	9.239 (9.438)	6.473 (6.351)	4.652 (4,580)	4.001 (4001)
5000		14.12 (14.90)	9.214 (9.399)	6.569 (6.325)	4.649 (4.566)	4.001 (3.999)
20,000				6.450 (6.302)	4.647 (4.552)	4.003 (3.992)

Table 7
Computed and predicted values of Nu_{1iHA}

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o					
	0.01	0.05	0.1	0.2	0.5	0.999
800		72.24 (72.41)	66.80 (68.2)	63.29 (65.3)	62.53 (64.5)	64.54 (64.6)
1000	**116.0** **(107.1)**	89.64 (88.33)	82.56 (83.3)	77.89 (79.7)	76.69 (78.0)	79.02 (79.0)
5000		**445.9** **(382.3)**	381.7 (361)	348.1 (346)	337.1 (340)	345.1 (345)
20,000			1275 (1252)	1221 (1229)	1247 (1249)	

Table 8
Comparison of predicted and computed values of α_{1i}

$(a_2^+ - a_2^+)_{w1}$	a_i/a_o						
	0.01	0.05	0.1	0.2	0.5	0.8	0.999
800		0.08569 (0.0898)	0.1060 (0.1046)	0.1303 (0.1256)	0.1694 (0.1676)	0.1974 (0.1980)	0.2153 (0.2151)
1000	**0.04762** **(0.0688)**	0.08071 (0.0877)	0.1014 (0.1021)	0.1265 (0.1223)	0.1658 (0.1639)	0.1937 (0.1937)	0.2115 (0.2105)
5000		**0.04128** **(0.0767)**	0.07534 (0.0895)	0.1067 (0.1078)	0.1470 (0.1441)	0.1720 (0.1704)	0.1874 (0.1852)
20,000			**0.05625** **(0.0821)**	0.09446 (0.0989)	0.1353 (0.1325)	0.1569 (0.1566)	0.1703 (0.1702)

molecular transport becomes dominant relative to turbulent transport, and $Pr_t \to \infty$.

From physical considerations, the correlating equations for α_o and α_i, as well as those for Nu_i and Nu_o, must converge to the same value as $a_o/a_i \to 1$. The decrease in Nu with decreasing a_i/a_o for inner heating and the opposite trend for outer heating, may be explained qualitatively on the basis of the increasing and decreasing areas for heat transfer, respectively, that is, in terms of the change in $2\pi rL$. Except in a few instances, the computed values of α_{1o} decrease monotonically with increasing values of Re and decrease monotonically with increasing values of a_i/a_o. The instances of non-monotonic and thereby somewhat questionable behavior, were ignored in the process of constructing the correlating equations.

Table 9
Computed and predicted values of $Nu_{1\mathrm{iHH}}$

$(a_\mathrm{o}^+ - a_\mathrm{i}^+)_\mathrm{wi}$	$a_\mathrm{i}/a_\mathrm{o}$					
	0.01	0.05	0.1	0.2	0.5	0.999
800		79.01 (79.85)	74.72 (76.2)	72.77 (74.7)	75.28 (76.7)	82.25 (82.3)
1000	**121.8 (115)**	97.51 (96.82)	91.88 (92.7)	89.17 (90.9)	91.93 (93.3)	100.2 (100.1)
5000		**456.1 (414.1)**	**412.8 (396)**	389.7 (388)	395.2 (397)	424.7 (423)
20,000				1408 (1390)	1412 (1416)	1503 (1506)

Table 10
Computed and predicted values of $Nu_{1\mathrm{iHC}}$

$(a_\mathrm{o}^+ - a_\mathrm{i}^+)_\mathrm{wi}$	$a_\mathrm{i}/a_\mathrm{o}$					
	0.01	0.05	0.1	0.2	0.5	0.999
800		66.54 (66.66)	60.40 (61.8)	55.99 (58.0)	53.48 (53.58)	53.11 (53.18)
1000	**110.7 (100)**	82.94 (81.5)	74.96 (75.5)	69.14 (71.1)	65.78 (65.74)	65.23 (65.27)
5000		428.2 (356)	355.0 (331)	314.5 (312.6)	293.9 (292)	290.6 (291)
20,000			**1367 (1205)**	1165 (1140)	1076 (1071)	1065 (1068)

Table 11
Computed and predicted values of $Nu_{0\mathrm{oHA}}$

$(a_\mathrm{o}^+ - a_\mathrm{i}^+)_\mathrm{wi}$	$a_\mathrm{i}/a_\mathrm{o}$					
	0.01	0.05	0.1	0.2	0.5	0.999
800		5.384 (5.376)	5.405 (5.405)	5.456 (5.460)	5.609 (5.605)	5.771 (5.771)
1000	5.381 (5.455)	5.403 (5.383)	5.423 (5.411)	5.471 (5.466)	5.643 (5.611)	5.783 (5.777)
5000		5.494 (5.424)	5.566 (5.453)	5.546 (5.507)	5.686 (5.854)	5.833 (5.828)
20,000			5.548 (5.486)	5.583 (5.541)	5.715 (5.688)	5.855 (5.856)

The computed values of $Nu_{1\mathrm{oHH}}$ for $a_\mathrm{i}/a_\mathrm{o} = 0.999$ were observed to be nearly equal to those of Yu et al. [12] for a uniformly heated round tube. This agreement is inexplicable on the basis of the integral formulation for Nu_1 but seems to be too good to be a mere coincidence. Such behavior is understandable for $Pr > Pr_\mathrm{t}$ in consideration of the identical dependence of Nu_∞^1 on Re for all shear flows in the equation of Churchill and Zajic, as is the failure of this agreement to extend to small values of Pr because Nu_0 for a round tube differs greatly from that for a parallel-plate channel. Since the characteristic length in Re and Nu is taken here to be the hydraulic diameter, it may be concluded from these observations that the hydraulic diameter concept is applicable as a good approximation for turbulent convection in annuli, including the limiting cases of a round

Table 12
Computed and predicted values of α_{0o}

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o					
	0.01	0.05	0.1	0.2	0.5	0.999
800		0.6923 (0.7032)	0.6830 (0.6869)	0.6562 (0.6561)	0.5659 (0.5715)	0.4430 (0.4432)
1000	0.6997 (0.7198)	0.6978 (0.7058)	0.6882 (0.6897)	0.6610 (0.6585)	0.5701 (0.5736)	0.4461 (0.4448)
5000		0.7221 (0.7234)	0.7104 (0.7065)	0.6820 (0.6748)	0.5877 (0.5878)	0.4589 (0.4555)
20,000			0.7198 (0.7205)	0.6914 (0.6881)	0.5955 (0.5993)	0.4643 (0.4647)

Table 13
Computed and predicted values of Nu_{0oHH}

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o					
	0.01	0.05	0.1	0.2	0.5	0.999
800	17.55 (18.84)	17.50 (18.11)	17.05 (17.25)	15.87 (15.84)	12.92 (13.08)	10.36 (10.36)
1000	17.92 (19.05)	17.88 (18.29)	17.39 (17.42)	16.14 (15.88)	13.08 (13.16)	10.44 (10.41)
5000	20.00 (20.58)	19.70 (19.61)	19.01 (18.58)	17.44 (16.94)	13.79 (13.71)	10.78 (10.68)
20,000	20.91 (22.10)	20.49 (20.88)	19.80 (19.63)	18.09 (17.76)	14.13 (14.20)	10.93 (10.94)

Table 14
Computed and predicted values of Nu_{0oHC}

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o					
	0.01	0.05	0.1	0.2	0.5	0.999
800	3.158 (3.117)	3.181 (3.159)	3.212 (3.204)	3.284 (3.297)	3.582 (3.556)	3.999 (3.998)
1000	3.116 (3.115)	3.183 (3.155)	3.212 (3.203)	3.294 (3.295)	3.582 (3.556)	3.999 (3.998)
5000	3.205 (3.106)	3.192 (3.177)	3.219 (3.195)	3.294 (3.298)	3.528 (3.560)	3.999 (3.998)
20,000	3.273 (3.099)	3.213 (3.140)	3.226 (3.189)	3.301 (3.282)	3.581 (3.557)	3.998 (3.998)

tube and a parallel-plate channel, only for values of $Pr \geqslant Pr_t$.

The near equality of the values of Nu for a parallel-plate channel and a round tube for $Pr \geqslant Pr_t$. suggests the possibility that, by analogy to $(u_m^+)_{wm}$ and despite the lack of a theoretical explanation in either case, a near-invariance might exist for intermediate values of a_i/a_o. The numerically computed values of Nu_{1oHH} in Table 17 do show some sign of convergence to the same value for $a_i/a_o \to 0$ as for $a_i/a_o = 1$, but the intermediate values are much greater. Because this comparison was necessarily for fixed values of $(a_o^+ - a_i^+)_{wi}$, supplementary numerical computations were carried out by iteration for a single fixed value of

Table 15
Computed and predicted values of $Nu_{1\mathrm{oHA}}$

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o					
	0.01	0.05	0.1	0.2	0.5	0.999
800	50.49 (50.34)	57.17 (57.98)	59.90 (60.16)	61.93 (61.17)	64.03 (63.60)	64.54 (64.62)
1000	61.14 (61.29)	69.98 (75.30)	73.20 (73.30)	75.77 (74.69)	78.33 (77.79)	79.01 (79.00)
5000	260.5 (275.4)	311.1 (305.2)	320.2 (318.5)	328.8 (324.5)	339.8 (338.2)	344.9 (344.7)
20,000	**879.8** **(1153)**	**1064** **(1111)**	1162 (1151)	1182 (1173)	1220 (1191)	1245 (1248)

Table 16
Computed and predicted values of α_{1o}

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o						
	0.01	0.05	0.1	0.2	0.5	0.8	0.999
800	**0.3738** **(0.4873)**	0.4555 (0.4610)	0.4326 (0.4387)	0.3940 (0.4001)	0.3034 (0.3059)	0.2440 (0.2381)	0.2155 (0.2152)
1000	**0.4205** **(0.4551)**	0.4609 (0.4502)	0.4163 (0.4288)	0.3830 (0.3915)	0.2975 (0.2992)	0.2397 (0.2329)	0.2115 (0.2106)
5000	**0.3502** **(0.4004)**	**0.2356** **(0.3940)**	**0.3139** **(0.3780)**	0.3255 (0.3438)	0.2635 (0.2631)	0.2123 (0.2050)	0.1877 (0.1853)
20,000	**0.6732** **(0.3679)**	0.3583 (0.3606)	**0.2345** **(0.3447)**	0.2867 (0.3156)	0.2413 (0.2412)	0.1934 (0.1883)	0.1706 (0.1703)

Table 17
Computed and predicted values of $Nu_{1\mathrm{oHH}}$

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o					
	0.01	0.05	0.1	0.2	0.5	0.999
800	**80.63** **(98.19)**	105.3 (107.5)	105.4 (107.1)	102.2 (102.1)	91.92 (89.11)	82.27 (82.34)
1000	**105.5** **(116.8)**	123.6 (128.6)	125.4 (128.5)	122.8 (122.8)	111.5 (107.9)	100.2 (100.2)
5000	**400.9** **(467.8)**	**407.0** **(504.9)**	**466.7** **(510.1)**	487.5 (494.4)	461.4 (446.5)	424.6 (423.1)
20,000	**2692** **(1839)**	**1658** **(1742)**	**1518** **(1756)**	**1567** **(1714)**	1608 (1571)	1501 (1504)

$Re = 37{,}600$. The monotonic increase in $Nu_{1\mathrm{oHH}}$ with decreasing values of a_i/a_o, as shown in Table 19, is very disappointing and implies that the near-coincidence of these limiting values has no obvious applicability for correlation or generalization for $0 < a_i/a_o < 1$.

The quantitative effect on Nu_{1i} of heat transfer to or from the outer surface, as calculated from the correlating equations $Nu_{1\mathrm{iHA}}$ and α_{1i}, is illustrated in Table 20 for $a_i/a_o = 0.2$ and 1.0. The values of Nu_{1i} for $-j_o a_o / j_i a_i = 0.05$, 0.10, 0.2, and 1.0 are intended to simulate the effect of heat losses of 5%, 10%, 20%, and 100%. It may be noted that the net rate of heating of the fluid, which is represented by Nu_{1i}, is reduced far less percentage-wise. Similar affects would be expected for other values of Pr.

Table 18
Computed and predicted values of Nu_{1oHC}

$(a_o^+ - a_i^+)_{wi}$	a_i/a_o					
	0.01	0.05	0.1	0.2	0.5	0.999
800	36.47 (33.84)	39.28 (39.68)	41.74 (41.81)	44.44 (43.67)	49.13 (47.36)	53.01 (53.17)
1000	43.05 (41.54)	48.80 (48.77)	51.68 (51.36)	54.78 (53.67)	60.37 (58.19)	65.21 (65.25)
5000	192.9 (195.1)	**251.8 (219.6)**	243.7 (231.4)	248.1 (241.4)	268.9 (260.4)	290.5 (290.5)
20,000	**525.8 (840.6)**	**783.4 (818.7)**	941.1 (855.6)	919.1 (891.8)	983.6 (959.2)	1064 (1067)

Table 19
Variation of Nu_{1oHH} with a_1/a_2 for Re = 37,600

a_i/a_o	0.01	0.05	0.1	0.2	0.5	0.8	0.9	0.999
Nu_{10HH}	152.3	144.8	141.0	132.3	114.2	104.6	102.2	100.2

Table 20
Effect of outer heat transfer on inner heat transfer as characterized by Nu_{1iHA}

$(a_o^+ - a_i^+)_{wi}$	α_{1i}	Nu_{1iHA}	Nu_{1iHH}	$-j_ia_i/j_oa_o$ = 1.0	0.2	0.1	0.05
a_i/a_o = 0.2							
800	0.1303	63.29	72.77	55.99	61.68	62.47	62.88
1000	0.1265	77.89	89.17	69.14	75.97	76.92	77.40
5000	0.1067	348.1	389.7	314.5	340.8	344.4	346.2
a_i/a_o = 1.0							
800	0.2154	64.54	82.26	53.1	61.87	63.18	63.85
1000	0.2115	79.02	100.2	65.22	75.81	77.38	78.19
5000	0.1876	345.0	424.7	290.5	332.5	338.6	341.8
20,000	0.1704	1246	1502	1055	1205	1225	1235

Table 21
Computed and predicted values of Nu_{0TT} and Nu_{1TT}

$2b^+$	1000	1600	2000	4000	10,000	20,000
$Nu_{0TT\text{-comp}}$	8.953	9.023	9.050	9.119	9.188	9.233
$Nu_{0TT\text{-pred}}$	8.952	9.007	9.032	9.100	9.180	9.233
$Nu_{1TT\text{-comp}}$	99.14	150.9	184.0	344.9	793.4	1496
$Nu_{1TT\text{-pred}}$	99.55	151.2	184.6	344.8	793.2	1496

Table 22
Predicted and numerically computed values of α_0 and α_1 for isothermally heated parallel plates

$2b^+$	1000	1600	2000	4000	10,000	20,000
Re	37,220	63,400	81,460	176,300	483,700	1,032,000
$\alpha_{0\text{-comp}}$	0.4498	0.4540	0.4555	0.4595	0.4634	0.4659
$\alpha_{0\text{-pred}}$	0.4498	0.4524	0.4537	0.4574	0.4623	0.4659
$\alpha_{1\text{-comp}}$	0.2281	0.2200	0.2186	0.2056	0.1926	0.1838
$\alpha_{1\text{-pred}}$	0.2283	0.2186	0.2146	0.2035	0.1915	0.1838

Table 23
Predicted and numerically computed values of Nu_{0TA} and Nu_{1TA}

$2b^+$	1000	1600	2000	4000	10,000	20,000
$Nu_{0TA\text{-comp}}$	4.926	4.927	4.928	4.929	4.930	4.031
$Nu_{0TA\text{-pred}}$	4.026	4.932	4.934	4.937	4.936	4.932
$Nu_{1TA\text{-comp}}$	76.53	117.7	144.5	274.0	640.6	1221
$Nu_{1TA\text{-pred}}$	76.83	118.1	145.0	274.6	641.4	1221

Table 24
Comparison of numerically predicted values of $Nu_{1(T_1-T_2)}$ and Nu_{1THC}

$2b^+$	Re_{4b}		$Nu_{1(T_1-T_2)}$	Nu_{1HC}
	Danov et al.	Herein	Danov et al.	Herein
1000	37,116	37,220	65.74	65.64
1600		63,400		101.3
2000	81,248	81,460	124.8	124.6
4000		176,300		237.3
20,000	1,029,440	1,032,000	1070	1069

2.8. Correlating equations for isothermal heating of parallel-plate channels

There are three thermal boundary conditions involving isothermal surfaces that result in fully developed convection in an annulus, firstly, uniform equal temperatures on the two surfaces, here designated by Nu_{TT}, secondly, a uniform temperature on one surface and perfect insulation on the other, here designated by Nu_{TA}, and thirdly, unequal uniform temperatures on the two surfaces, here designated as Nu_{HC}. As implied by this notation, the latter condition results in a uniform heat flux across the channel and no longitudinal development in the temperature of the fluid. As such, it has already been treated. (See the values for a_i/a_o in Tables 6, 10, 14, and 18.) On the other hand, the first and second conditions both result in a longitudinal development in the temperature of the fluid and can be related to one another through an influence coefficient. This influence coefficient is an alternative variable for correlation in place of Nu_{TA} or Nu_{TT}, just as with uniform heating, but, by contrast has no other utility.

The numerical computations for Nu_{TA} and Nu_{TT} require iteration because the integrand of the integral expression for the total local heat flux density incorporates the temperature distribution. As a result of the greatly increased computational demands as well as the very limited practical interest in these conditions, the numerical computations for these two quantities were limited to a parallel-plate channel.

Rather than devise correlating equations for Nu_{0TA} and Nu_{1TA} directly, as was done for their counterparts for uniform heating, correlating equations were instead, in the interests of illustrating an alternative approach, devised for Nu_{0TT} and Nu_{1TT}, and for the values of α_0 and α_1 as determined from the numerically computed values of Nu_{0TA} and Nu_{0TT}, and Nu_{1TA} and Nu_{1TT}, respectively. These four expressions are included in Table 1. The "asymptotic" value of 12 for the hypothetical case of $u_m^+ \to \infty$ in the correlating equation for Nu_{0TT} may be recognized as the theoretical value for fully developed convection in plug flow in a uniformly heated parallel-plate channel, but the dependence on u_m^+ is purely empirical. (The corresponding value of 8, which may be recognized as solution for convection in plug flow in a uniformly round tube, would have appeared in the correlating for Nu_{0oHH} if it had been developed directly.) These two values are artifacts of the integral formulations for Nu_0 just as $Re(f/2)$ is for Nu_1. The predictions of the correlating equations and the numerically computed values for α_0, α_1, Nu_{0TT}, and Nu_{1TT} are observed in Tables 22 and 23 to be in almost perfect agreement, as are those for Nu_{0TA} and Nu_{1TA}, as determined by superposition, in Table 24. A reexamination of the predictions in Tables 3–18 for $a_i/a_o = 0.999$ reveals similar agreement and indicates that it only the dependence on a_1/a_2 that is predicted somewhat uncertainly.

As mentioned previously, Nu_{1HC} (for equal uniform heating and cooling) and $Nu_{1(T_1-T_2)}$ (for heat transfer between isothermal surfaces at different temperatures) are equivalent on conceptual grounds. This equality is confirmed in an operational sense in Table 24 in which the values of $Nu_{1(T_1-T_2)}$ computed by Danov et al. [22] are seen to agree within round-off error with computed values of Nu_{1HC} from the current investigation. Since

$Nu_{0(T_1-T_2)}$ and $Nu_{0\mathrm{THC}}$ are equal to exactly 4.0, the value for pure thermal conduction across a slab of width $2b$, a correlating equation is unnecessary for either Nu or α.

3. Summary and conclusions

Predictive equations for *Nu* for both surfaces have been developed for all combinations of uniform heating and cooling in circular concentric annuli of all aspect ratios for all values of *Re* in the fully turbulent regime and all values of *Pr*. Predictive equations are also included for parallel-plate channels with all combinations of uniform wall temperature. These predictive equations are summarized in Tables 1 and 2. No further information is required for their quantitative application. They are called predictive in that they were constructed without direct reference to experimental data. These expressions appear somewhat formidable at first glance, but this a necessary price for the inclusion of so many independent variables and parameters.

Tables 3–24 provide tests of the limits of applicability of the predictive equations and their accuracy within these limits. The numerical calculations of this overall investigation and prior ones rather than experimental data are used as a criterion. However, the numerical calculations of this overall investigation have already been validated by comparisons with experimental data in Yu et al. [2,3].

The success in devising such comprehensive, accurate, and relatively simple predictive equations is a consequence of utilizing the generalized correlating equation of Churchill amd Usagi [7], the analogies devised by Churchill and Zajic [16], and the principle of superposition, as validated herein for turbulent convection in annuli.

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